A study of the combined effect of thermal radiative transfer and rotation on the gravitational stability of a hot fluid

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The effect of radiative transfer on thermal stability when a Coriolis force is also acting has been examined. Two asymptotic approximations of the radiative transfer equation have been used to study the stability problem in detail. The necessary and sufficient conditions for the validity of the principle of exchange of stabilities are also obtained.

1. Introduction

The theoretical explanation of the experimental observations of Bénard on the behaviour of a fluid enclosed between two plates and heated from below, first formulated and solved as a stability problem by Rayleigh (1916), has been the subject of similar investigations later by several authors, notably Pellew & Southwell (1940) and Chandrasekhar (1961). The latter dealt, at length, in a series of papers with its various ramifications, for example, when the fluid is ionized and is in the presence of a magnetic field, or when it is rotating, etc. Goody (1956) and Spiegel (1960) estimated the radiative transfer effects in the original problem solved by Rayleigh or Pellew & Southwell. More recently we (1962, hereafter referred to as I) extended these results (namely, radiative transfer effects) to the case of an ionized medium in the presence of a vertical magnetic field. If rotation be added to such a system one would be dealing with a problem of very great generality applicable to both astrophysical and terrestrial contexts. However, the number of independent parameters becomes rather large. The classical problem of Rayleigh with radiative transfer and rotation alone is of sufficient interest to warrant a separate formulation because of its interest in the atmosphere of the planets, especially the earth. The present paper is devoted to such a study.

§§ 2 and 3 are devoted to the derivation of the basic equations and the equations for marginal stability. In § 4 a variational principle is established and it is used in § 5 to obtain the critical value of the Rayleigh number for the onset of convection. The discussion of the principle of exchange of stabilities, and the manner of onset of instability constitutes §§ 6 and 7. The analysis is confined to the case when both bounding surfaces are free. When they are rigid (or one rigid and the other free), the variable nature of the temperature gradient, because of radiative transfer, in the equilibrium state, does not permit a variational formulation of the problem.

2. Basic equations

The basic equations for the investigation of the radiative transfer effects on the convective instability of a hot rotating fluid enclosed between two plates are the equations of hydrodynamics referred to a rotating system of co-ordinates along with the integro-differential equation for the radiative transfer. These in vector form are

$$\partial\rho/\partial t + \nabla.(\rho\mathbf{u}) = 0, \tag{1}$$

$$\rho[\partial\mathbf{u}/\partial t + (\mathbf{u}.\nabla)\mathbf{u}] = -\nabla(P - \tfrac{1}{2}|\mathbf{\Omega}\times\mathbf{r}|^2) - g\rho\boldsymbol{\lambda} + \rho\nu\nabla^2\mathbf{u} - 2\rho\mathbf{\Omega}\times\mathbf{u}, \tag{2}$$

$$\partial T^*/\partial t + (\mathbf{u}.\nabla)T^* = \Phi/c_p + \kappa\nabla^2 T^*, \tag{3}$$

$$dI/ds = k[B^* - I(\mathbf{s})], \tag{4}$$

$$\Phi = -\int\frac{dI}{ds}\,d\omega = -4\pi kB^* + k\int I\,d\omega, \tag{5}$$

$$\rho = \rho_0(1-\alpha T^*). \tag{6}$$

In these equations $\mathbf{\Omega}$ is the angular velocity about a vertical axis specified by a unit vector $\boldsymbol{\lambda} = (0, 0, 1)$, $\mathbf{r}$ the position vector of any point, $\mathbf{u}$ the vector velocity, ρ the density, P the pressure, ν the kinematic viscosity, g the gravitational acceleration, T^* temperature, κ the thermal diffusivity, c_p and Φ the specific heat and the radiative heating per unit volume, I the intensity of radiation at any point, k the absorption coefficient, B^* the Planck function, ω an element of solid angle and $\mathbf{s}$ an element of vector length. As in I, and as usually done in problems of this kind, we split up T^*, Φ and P into $(T_0+\theta)$, $(\Phi_0+\phi)$ and (P_0+p), respectively, where the quantities with subscript '0' now refer to the equilibrium or the static state and p and ϕ are the perturbations due to temperature rise θ. $\mathbf{u}$ in this context is now the disturbance velocity. The perturbations p, ϕ, θ and $\mathbf{u}$ are assumed to be small and hence satisfy a linearized version of the equations (1) to (6). The equations for the static case are the pressure and temperature distributions given by

$$d(P_0 - \tfrac{1}{2}|\mathbf{\Omega}\times\mathbf{r}|^2)/dz = -\rho_0 g[1 - \alpha\textstyle\int\beta\,dz], \tag{7}$$

$$\kappa(d^2T_0/dz^2) + \Phi_0/c_p = 0, \tag{8}$$

where β is the vertical temperature gradient dT_0/dz and is to be obtained from the solution of (8). This, as obtained by Goody (1956), is quoted later in the paper. The linearized system of equations is

$$\partial\mathbf{u}/\partial t = -(1/\rho_0)\nabla p + \nu\nabla^2\mathbf{u} + \gamma\theta\boldsymbol{\lambda} - 2\mathbf{\Omega}\times\mathbf{u}, \tag{9}$$

$$\partial\theta/\partial t = \phi/c_p - \beta w + \kappa\nabla^2\theta, \tag{10}$$

$$\operatorname{div}\mathbf{u} = 0, \tag{11}$$

where $\gamma = g\alpha$. Now taking the curl and curl curl of (9), we obtain, in terms of the vertical components after making use of (11),

$$\partial\zeta/\partial t = \nu\nabla^2\zeta + 2\Omega(\partial w/\partial z) \tag{12}$$

and

$$-\partial(\nabla^2 w)/\partial t = -\gamma\nabla_1^2\theta - \nu\nabla^4 w + 2\Omega(\partial\zeta/\partial z), \tag{13}$$

where ζ and w are the vertical components of the vorticity and velocity vectors respectively and $\nabla_1^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$. From now onward we shall be dealing with equations (10), (12) and (13).

The boundary conditions are not affected by radiation and remain the same as those for Chandrasekhar's (1953) corresponding case. They are, for free bounding surfaces,

$$\left.\begin{array}{llll} \theta = 0, & w = 0 & \text{for} \quad z = 0 & \text{and} \quad z = d, \\ \partial^2 w/\partial z^2 = 0, & \partial\zeta/\partial z = 0 & \text{for} \quad z = 0 & \text{and} \quad z = d. \end{array}\right\} \tag{14}$$

3. The equations for the case of marginal stability

The physical conditions for the validity of the principle of exchange of stabilities are derived later. In this section we discuss the marginal stability. It is characterized by $\partial/\partial t = 0$. The equations (10), (12) and (13) then become

$$\nu\nabla^4 w - 2\Omega(\partial\zeta/\partial z) = -\gamma\nabla_1^2\theta, \tag{15}$$

$$\nabla^2\zeta = -(2\Omega/\nu)\,\partial w/\partial z, \tag{16}$$

$$\beta w = \kappa\nabla^2\theta + \phi/c_p. \tag{17}$$

We will solve these equations for the two asymptotic approximations of ϕ, which are given in I. After putting ϕ in terms of θ and then eliminating θ between (15), (16) and (17), we get for the two cases

$$\nabla^6 w = \frac{4\pi k S^*}{\kappa c_p}\nabla^4 w - \frac{4\Omega^2}{\nu^2}\frac{\partial^2 w}{\partial z^2} - \frac{4\pi k S^*}{\kappa c_p}\frac{2\Omega}{\nu}\frac{\partial\zeta}{\partial z} - \frac{\gamma}{\nu\kappa}\nabla_1^2\beta w$$

$$(k^2d^2 \ll a^2) \quad \text{(called case } (a) \text{ hereafter)}, \tag{18}$$

$$\left(1 + \frac{4\pi S^*}{3\kappa k c_p}\right)\nabla^6 w = -\frac{4\Omega^2}{\nu^2}\left(1 + \frac{4\pi S^*}{3\kappa k c_p}\right)\frac{\partial^2 w}{\partial z^2} - \frac{\gamma}{\nu\kappa}\nabla_1^2\beta w$$

$$(k^2d^2 \gg a^2) \quad \text{(called case } (b) \text{ hereafter)}, \tag{19}$$

where $S^* = 4\pi\sigma_0(T_0 + \theta)^3$ is assumed to be a constant, σ_0 being the Stefan constant. We solve the above equations by the method of separation of variables by putting $w = f(x^*, y^*)\,W(z)$ where

$$\nabla_1^2 f = -(a^2/d^2) f. \tag{20}$$

The number 'a' characterizes the cell shape and size. Defining a dimensionless variable $\xi = (z/d) - \frac{1}{2}$, we have

$$\left.\begin{array}{l} \dfrac{\partial^2}{\partial z^2} = \dfrac{1}{d^2}\dfrac{\partial^2}{\partial \xi^2} = \dfrac{D^2}{d^2}, \\ \nabla^2 = \nabla_1^2 + \partial^2/\partial z^2 = (D^2 - a^2)/d^2. \end{array}\right\} \tag{21}$$

Utilizing (20) and (21), equations (18) and (19) can be written as

$$[(D^2 - a^2)^3 + TD^2]\,w = 3k^2d^2\chi(D^2 - a^2)\,w - 3k^2d^2\chi(2\Omega d^3/\nu)\,D\zeta$$

$$-Ra^2(\beta/\bar\beta)\,w \quad (k^2d^2 \ll a^2) \tag{22}$$

and

$$[(D^2 - a^2)^3 + TD^2]\,(1 + \chi)\,w = -Ra^2(\beta/\bar\beta)\,w \quad (k^2d^2 \gg a^2), \tag{23}$$

where $\chi = 4\pi S^*/3\kappa k c_p$, $T = 4\Omega^2 d^4/\nu^2$ and $R = \gamma\bar\beta d^4/\nu\kappa$ is the Rayleigh number, T the Taylor number, $\bar\beta$ being the average temperature gradient. Also equation (16) can be written as

$$(D^2 - a^2)\,\zeta = -(2\Omega d/\nu)\,Dw. \tag{24}$$

The boundary conditions (14) reduce to

$$\left.\begin{aligned} w = 0, \quad D\zeta = 0, \\ D^2w = D^4w = 0, \end{aligned}\right\} \quad \text{for} \quad \xi = \pm\tfrac{1}{2}. \tag{25}$$

Goody (1956) has given the following solution for $\beta/\bar{\beta}$ in the static case, namely

$$\beta/\bar{\beta} = L\cosh\lambda\zeta + M, \tag{26}$$

L and M being constants given by

$$L = \chi[(2\chi/\lambda)\sinh\tfrac{1}{2}\lambda + \tfrac{1}{2}(3+3\chi)^{\frac{1}{2}}\sinh\tfrac{1}{2}\lambda + \cosh\tfrac{1}{2}\lambda]^{-1},$$

$$M = (L/\chi)[\tfrac{1}{2}(3+3\chi)^{\frac{1}{2}}\sinh\tfrac{1}{2}\lambda + \cosh\tfrac{1}{2}\lambda],$$

and

$$\lambda^2 = 3k^2d^2(1+\chi).$$

Equations (22) to (26) completely determine the marginal state for the two asymptotic cases. In the next section a variational principle for the problem at hand will be established.

4. A variational principle

First, R will be expressed as the ratio of two positive definite integrals and then it will be used to show that R so obtained will be minimum provided the differential equation for w is satisfied. The variational principle will be established for both the asymptotic approximations mentioned earlier. It may be pointed out here that this variational principle, which represents an extension of the one given by Malkus (1954), is only true when the bounding surfaces are free. The stationary property of R for the case of both rigid (or one rigid and the other free) surfaces cannot be established.

Case (*a*). Multiplying equation (24) on both sides by ζ and integrating from $-\frac{1}{2}$ to $\frac{1}{2}$, we get

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} [(D\zeta)^2 + a^2\zeta^2]\, d\xi = -\frac{2\Omega d}{\nu}\int_{-\frac{1}{2}}^{\frac{1}{2}} wD\zeta\, d\xi. \tag{27}$$

Also multiplying on both sides of (22) by w and integrating from $-\frac{1}{2}$ to $\frac{1}{2}$, we have after utilizing (27)

$$R = (I_1 + 3k^2d^2\chi I_2 + TI_3 + 3k^2d^4\chi I_4)/a^2I_5 = I/a^2I_5, \tag{28}$$

where

$$I_1 = \int_{-\frac{1}{2}}^{\frac{1}{2}} [(D^3w)^2 + 3a^2(D^2w)^2 + 3a^4(Dw)^2 + a^6w^2]\, d\xi,$$

$$I_2 = \int_{-\frac{1}{2}}^{\frac{1}{2}} [(D^2w)^2 + 2a^2(Dw)^2 + a^4w^2]\, d\xi, \quad I_3 = \int_{-\frac{1}{2}}^{\frac{1}{2}} (Dw)^2\, d\xi,$$

$$I_4 = \int_{-\frac{1}{2}}^{\frac{1}{2}} [(D\zeta)^2 + a^2\zeta^2]\, d\xi, \quad I_5 = \int_{-\frac{1}{2}}^{\frac{1}{2}} (\beta/\bar{\beta}w^2)\, d\xi.$$

Consider now the effect on R of an arbitrary variation in w compatible with the boundary conditions (25). We have

$$\delta R = (a^2I_5)^{-1}[\delta I_1 + 3k^2d^2\chi\delta I_2 + T\delta I_3 + 3k^2d^4\chi\delta I_4 - (I/I_5)\,\delta I_5], \tag{29}$$

δI_i denoting the corresponding variation in I_i. Also from (27), we have

$$-2\int_{-\frac{1}{2}}^{\frac{1}{2}}\delta\zeta\,(D^2-a^2)\,\zeta d\xi = \frac{2\Omega d}{\nu}\left[\int_{-\frac{1}{2}}^{\frac{1}{2}}\delta\zeta Dw d\xi-\int_{-\frac{1}{2}}^{\frac{1}{2}}\delta w D\zeta d\xi\right]$$

or $$\int_{-\frac{1}{2}}^{\frac{1}{2}}\delta\zeta(D^2-a^2)\,\zeta d\xi = -\int_{-\frac{1}{2}}^{\frac{1}{2}}\delta\zeta\left\{(D^2-a^2)\,\zeta+\frac{2\Omega d}{\nu}Dw\right\}d\xi+\frac{2\Omega d}{\nu}\int_{-\frac{1}{2}}^{\frac{1}{2}}\delta w D\zeta d\xi. \tag{30}$$

Now from (28), integrating by parts and making use of (30) and boundary conditions, we get

$$\delta R = -\frac{2}{a^2 I_5}\left[\int_{-\frac{1}{2}}^{\frac{1}{2}}\delta w\left\{(D^2-a^2)^3 w-3k^2d^2\chi(D^2-a^2)^2 w+TD^2w\right.\right.$$
$$\left.+3k^2d^2\chi\frac{2\Omega d^3}{\nu}D\zeta+Ra^2\beta/\bar\beta\, w\right\}d\xi$$
$$\left.-3k^2d^4\chi\left(\int_{-\frac{1}{2}}^{\frac{1}{2}}\delta\zeta\left\{(D^2-a^2)\,\zeta+\frac{2\Omega d}{\nu}Dw\right\}d\xi\right)\right].$$

Now $\delta R \equiv 0$ for all small arbitrary variations δw, $\delta\zeta$ in w and ζ respectively, provided equations (22) and (24) are satisfied. Its converse is also true.

Case (*b*). The corresponding expression for R in this case can easily be seen to be

$$R = \frac{1+\chi}{a^2}\frac{I_1+TI_3}{I_5}. \tag{31}$$

The same kind of analysis can be extended to prove that the expression given by (23) is a minimum. Now a variational procedure analogous to that enunciated in I can be used to obtain the value of the critical Rayleigh number in the present case.

5. Determination of R_c

The critical Rayleigh number for the onset of convection can be calculated by choosing the trial function satisfying the boundary conditions as

$$w = \text{const.}\sin m\pi(\xi+\tfrac{1}{2}). \tag{32}$$

As distinguished from other free-boundary problems, this trial function does not represent the exact solution of the differential equations in the present case because of the variation of β, which in turn is due to radiative transfer. The latter, however, does not effect the boundary conditions, so that we can still choose the trial function so often used in similar problems. Substituting (32) in (24), we obtain

$$\zeta = \text{const.}\frac{2\Omega d}{\nu}\frac{m\pi}{m^2\pi^2+a^2}\cos m\pi(\xi+\tfrac{1}{2}), \tag{33}$$

m being an integer. Substituting (32) and (33) in (28) we obtain after integration, for case (*a*),

$$R = \frac{(m^2\pi^2+a^2+3k^2d^2\chi)\,[(m^2\pi^2+a^2)^3+Tm^2\pi^2]}{a^2(m^2\pi^2+a^2)\,[M+8m^2\pi^2\sinh\frac{1}{2}\lambda L/\lambda(\lambda^2+4m^2\pi^2)]}. \tag{34}$$

Let $a^2 = m^2\pi^2 x$. Then (34) becomes

$$R = \frac{m^4\pi^4}{D_1 x}\frac{(1+x+3k^2d^2\chi_1)}{(1+x)}[(1+x)^3+T_1], \tag{35}$$

where $\chi_1 = \chi/\pi^2$, $T_1 = T/\pi^4$ and

$$D_1 = \left[M + \frac{8m^2\pi^2 \sinh\frac{1}{2}\lambda L}{\lambda(\lambda^2+4m^2\pi^2)}\right].$$

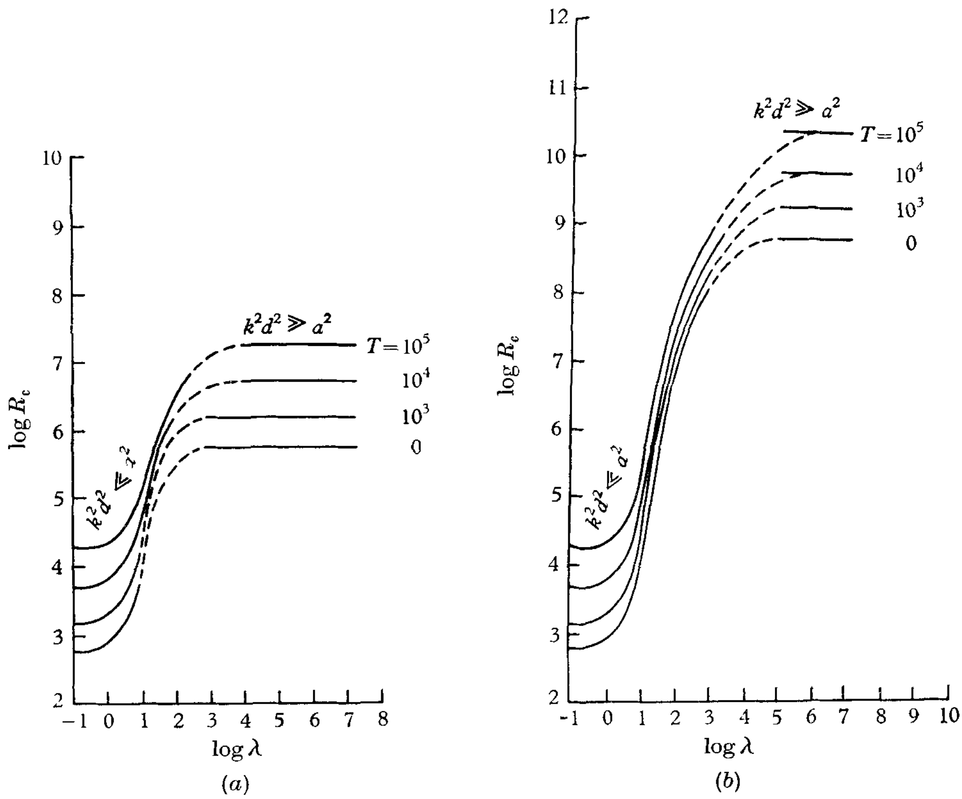

FIGURE 1. Plot of critical Rayleigh number R_c as a function of λ for the different values of Taylor number T given on the curves. The dimensionless quantities λ, χ and T are characteristic of the absorption coefficient and the distance between the horizontal planes, temperature in the equilibrium state and rotation respectively. The dotted lines joining the full ones represent the interpolation of the two approximations where neither of them holds. (a) $\chi = 10^3$, (b) $\chi = 10^6$.

For a given x, instability will first set in for the lowest mode $m = 1$. Thus, we get

$$R = \frac{\pi^4}{D_1 x}\frac{1+x+3k^2d^2\chi_1}{1+x}[(1+x)^3+T_1]. \tag{36}$$

For R to be minimum $dR/dx = 0$, i.e.

$$2x^5 + (7+3k^2d^2\chi_1)x^4 + 2(4+3k^2d^2\chi_1)x^3 + (2-T_1)x^2 - 2(1+3k^2d^2\chi_1)(1+T_1)x - (1+3k^2d^2\chi_1)(1+T_1) = 0. \tag{37}$$

k^2d^2	λ \ T	10^3		10^4		10^5	
		a^2	R_c/R_{c0}	a^2	R_c/R_{c0}	a^2	R_b/R_{c0}
3.33×10^{-6}	10^{-1}	1.3819×10	2.5510	3.2471×10	8.1822	7.4417×10	3.2422×10
3.33×10^{-4}	1	1.4045×10	2.7222	3.2847×10	8.5793	7.4841×10	3.3612×10
3.33×10^{-2}	10	2.0776×10	2.9950×10	4.4725×10	7.1944×10	9.4028×10	1.7980×10^2
3.33×10^{0}	10^2	—	—	5.4021×10	2.7087×10^3	1.2009×10^2	5.3372×10^3
3.33×10^{2}	10^3	—	2.5516×10^3	—	—	—	—
3.33×10^{4}	10^4	—	2.5516×10^3	—	8.1860×10^3	—	3.2443×10^4
3.33×10^{6}	10^5	—	2.5516×10^3	—	8.1860×10^3	—	3.2443×10^4
3.33×10^{8}	10^6	—	2.5516×10^3	—	8.1860×10^3	—	3.2443×10^4
3.33×10^{10}	10^7	—	2.5516×10^3	—	8.1860×10^3	—	3.2443×10

TABLE 1. The values of k^2d^2, a^2, R_c/R_{c0} for different T's and for $\chi = 10^3$.

k^2d^2	λ \ T	10^3		10^4		10^5	
		a^2	R_c/R_{c0}	a^2	R_c/R_{c0}	a^2	R_c/R_{c0}
3.33×10^{-9}	10^{-1}	1.3768×10	2.5511	3.2471×10	8.1823	7.4417×10	3.2422×10
3.33×10^{-7}	1	1.4054×10	2.7229	3.2848×10	8.5816	7.4845×10	3.3626×10
3.33×10^{-5}	10	2.0854×10	4.2318×10	4.4732×10	8.8028×10	9.4024×10	2.3403×10^2
3.33×10^{-3}	10^2	2.4259×10	2.0155×10^4	5.4028×10	3.3408×10^4	1.2008×10^2	6.9216×10^4
3.33×10^{-1}	10^3	2.4321×10	3.0132×10^5	5.4221×10	5.2358×10^5	1.2091×10^2	1.0246×10^6
3.33×10	10^4	—	—	—	—	—	—
3.33×10^{3}	10^5	—	2.5490×10^6	—	8.1778×10^6	—	3.2410×10^7
3.33×10^{5}	10^6	—	2.5490×10^6	—	8.1778×10^6	—	3.2410×10^7
3.33×10^{7}	10^7	—	2.5490×10^6	—	8.1778×10^6	—	3.2410×10^7

The value of the Rayleigh number as given by Pellew and Southwell with no rotation and radiation is equal to 657·5.

TABLE 2. The values of k^2d^2, a^2, R_c/R_{c0} for different T's and for $\chi = 10^6$.

From (37) it is easily seen that there cannot be more than one change in sign for different values of the parametric coefficients, i.e. there will be only one positive root of this equation. This positive root when substituted in (36) will give us the critical value of R.

For case (*b*) the equations corresponding to (36) and (37) are

$$R = (1+\chi)\pi^4[(1+x)^3+T_1]/xD_1 \tag{38}$$

and

$$2x^3+3x^2 = 1+T_1. \tag{39}$$

The value of x determined as the root of (39) gives on substitution in (38)

$$R_c = \{(1+\chi)/D_1\}R_{c0}, \tag{40}$$

where R_{c0} is the critical value of the Rayleigh number with rotation and no radiation and is the same as obtained by Chandrasekhar (1953). Figures 1 (*a*) and (*b*) show the plot of $\log R_c$ *vs* $\log \lambda$ for $\chi = 10^3, 10^6$ and for the different values of T shown on the curves (in the Earth's atmosphere $\chi \sim 10^5$). Tables 1 and 2 show the values of a^2 and R_c/R_{c0} for different T and λ and for $\chi = 10^3, 10^6$.

6. The principle of exchange of stabilities

Now we shall examine the principle of exchange of stabilities and the concept of over-stability for the problem under consideration. The basic equations for the present case are (10), (12) and (13). If the time dependence is assumed to be like $\exp(p^*t)$, then the above equations become

$$\left.\begin{aligned}(p^*-\nu\nabla^4)\nabla^2 w &= -(\gamma a^2/d^2)\theta - 2\Omega(\partial\zeta/\partial z),\\ (p^*-\nu\nabla^2)\zeta &= 2\Omega(\partial w/\partial z),\\ (p^*-\kappa\nabla^2)\theta &= \phi/c_p - \beta w.\end{aligned}\right\} \tag{41}$$

Now following the arguments and different mathematical steps given in I, it can be shown that sufficient conditions for the validity of the principle of exchange of stabilities are

Case (*a*):

$$\left.\begin{aligned}\nu &> \kappa, & \text{(i)},\\ \int_{-\frac{1}{2}}^{\frac{1}{2}}|Dw|^2 d\xi &> d^2\int_{-\frac{1}{2}}^{\frac{1}{2}}|\zeta|^2 d\xi & \text{(ii)}.\end{aligned}\right\} \tag{42}$$

Case (*b*):

$$\nu > \kappa_1, \quad \text{where} \quad \kappa_1 = \kappa(1+\chi). \tag{43}$$

In the limiting case when the fluid does not absorb or emit radiation, i.e. when χ or $\lambda \to 0$, equations (42) and (43) reduce to Chandrasekhar's (1953) result, given by (42). Equation (42, ii) appears in its present form, along with (42, i) simultaneously, due to the presence of the radiation term in the above equations. It is, however, independent of χ and λ. This leads one to think that this may also be true in the absence of radiative transfer. That this is the case can easily be shown by employing the method given by Pellew & Southwell for the equations of the classical problem, although not hitherto done. Hence this would imply that this condition has also to be satisfied along with (43) for the case (*b*). Strictly speaking, therefore, the results of Chandrasekhar (1953), as also those based on this investigation, will be true only when this condition, viz. (42, ii), is satisfied along with

(42, i). It is difficult to predict the results within the framework of this analysis, when one of them is not satisfied. Physically speaking, the condition (42, ii) seems to put an upper limit on the value of Ω or T. When (42) or (43) is violated we have the case of over-stability, the Rayleigh number for which can easily be obtained through a succession of steps as explained in I.

Case (a).

$$R = \frac{\pi^4}{D_1}\frac{2(\nu+\kappa)}{\kappa}\frac{1}{x}\left[(1+x)^3 + 3k^2d^2\frac{\chi_1\kappa}{(\nu+\kappa)}(1+x)^2 + \frac{T_1\nu^2}{(\nu+\kappa)^2[1+3k^2d^2\chi_1\kappa/\{(\nu+\kappa)(1+x)\}]}\right], \tag{44}$$

where $\chi_1 = \chi/\pi^2$, $T_1 = T/\pi^4$ and the equation which determines $x_{\min}$ is

$$\frac{(1+x)(1+x+B)^2}{(1+x)^2+B}[2x^2+(1+B)x-(1+B)] = \frac{T_1\nu^2}{(\nu+\kappa)^2}, \tag{45}$$

where $$B = 3k^2d^2\chi_1\kappa/(\nu+\kappa).$$

Case (b). The corresponding expressions for R and x are

$$R = \frac{\pi^4(1+\chi)}{D_1}\frac{2(\nu+\kappa_1)}{\kappa_1}\frac{1}{x}\left[(1+x)^3+\frac{T_1\nu^2}{(\nu+\kappa_1)^2}\right] \tag{46}$$

and $$2x^3+3x^2 = 1+T_1\nu^2/(\nu+\kappa_1)^2. \tag{47}$$

In the limit when χ or $\lambda \to 0$, equations (44) to (47) reduce to Chandrasekhar's (1953) results.

7. Manner of onset of instability

In order to know which type of instability will arise first, we study the limiting behaviour of the critical Rayleigh numbers for large values of T in the two cases. This will also provide us with the necessary condition for the validity of the principle of exchange of stabilities.

Let $R_c^{(\text{con})}$ and $R_c^{(\text{o.s.})}$ denote the limiting values of the critical Rayleigh numbers evaluated from (36) and (44) respectively for $T \to \infty$.

Case (a). From (37), we have

$$x_{\min} \to (\tfrac{1}{2}T_1)^{\frac{1}{3}} \tag{48}$$

and $$R_c^{(\text{con})} \to (3\pi^4/D_1)(\tfrac{1}{2}T_1)^{\frac{2}{3}}. \tag{49}$$

Again from (44) and (45), we get

$$R_c^{(\text{o.s.})} \to (3\pi^4/D_1)\,2\nu^{\frac{4}{3}}(\tfrac{1}{2}T_1)^{\frac{2}{3}}/\kappa(\nu+\kappa)^{\frac{1}{3}}. \tag{50}$$

Now suppose instability as convection arises earlier than over-stability, then $R_c^{(\text{con})} < R_c^{(\text{o.s.})}$, or utilizing (49) and (50), we get

$$(\kappa/\nu)(1+\kappa/\nu)^{\frac{1}{3}} < 2. \tag{51}$$

If we denote by $(\kappa/\nu)^*$ the value of (κ/ν) which makes (51) an equality, then the condition for convection to arise first becomes

$$\kappa/\nu < (\kappa/\nu)^*. \tag{52}$$

This is also the equation obtained by Chandrasekhar (1953) in the absence of radiative transfer. Thus radiation does not affect the condition under which convection will arise or the necessary condition for the validity of the principle of exchange of stabilities.

Case (b). For this approximation, $x_{\min}$ in the case of convection as well as over-stability varies as $(T_1)^{\frac{1}{3}}$ in the limit $T \to \infty$. But this clearly violates the restriction $k^2d^2 \gg a^2$ essential for this approximation. Thus like the magnetic field, large rotation also lends a transparent character to the fluid.

8. Discussion

Rotation and radiative transfer both have an inhibiting influence on the thermal instability of the fluid. In addition to the results similar to those obtained in I, there is a striking difference in the manner of the onset of instability under the first approximation. While in the presence of a magnetic field over-stability arises earlier than convection for large values of λ this is not the case in the present problem. If the necessary condition for the validity of the principle of exchange of stability, as given by Chandrasekhar (1953) for no radiation, is satisfied then the only mode of instability which the fluid is capable of is by convection for large values of λ (within the first approximation) even. The necessary condition (52) as compared to the corresponding condition in the presence of a magnetic field does not in any way depend upon λ. Furthermore, it is observed while discussing the principle of exchange of stabilities that the two conditions (42) for its validity which are obtained by applying two different methods for the classical problem appear simultaneously in the presence of radiation. In this manner the behaviour of the radiation term in the equations is quite peculiar in bringing out the facts which are otherwise implicit in the equations.

The reason for the agreement of the theoretical investigations of Chandrasekhar with the experimental results of Nakagawa, Fultz and others (see Chandrasekhar 1961) is that the temperatures involved in the experiments were not high enough and thus the only mechanism which could substantially contribute to the vertical transport of heat was thermal conduction. Under these circumstances the temperature gradient in the equilibrium state could not deviate much from constancy which was among the most important assumptions of all the theoretical investigations. As is evident from the results of §5 the variable temperature gradient which has its origin in the radiative transfer process contributes much to the stability through D_1.

In view of the fact that the onset of convection in the present case does not in any way depend upon λ, while it does so for the case presented in I, it is of interest to investigate the manner of onset of instability of a hot, radiant and electrically conducting fluid when a vertical magnetic field is acting along with rotation. This presents a large amount of numerical work. The results of this investigation we hope to publish sometime in the future.

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